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Attenuating oscillations in uncertain dynamic systems

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Abstract. Two basic approaches exist for the design of robust H_∞-controllers used to optimally attenuate oscillations in uncertain dynamic systems. One of these is based on solving Riccati equations; the other approach involves a linear matrix-inequality (LMI) technique. It is shown that the Riccati equations associated with this problem, which contain additional parameters (scalings) as Lagrangian multipliers, are feasible only when the values of these parameters are within a parallelepiped whose boundaries are to be determined. A new algorithm for synthesizing a robust H_∞-controller, using the LMI technique, is suggested. The boundaries of the admissible values of the scalings are identified. An illustrative example is considered, which concerns the optimal attenuation of oscillations of a parametrically disturbed pendulum.

Key words: linear matrix inequality, robust H_∞-control, uncertain dynamic system

1. Introduction

Within the framework of the state-space technique, there are two main approaches to H_∞-controller design. One of these is based on solving an algebraic Riccati equation [1], while the other involves linear matrix inequalities (LMIs) [2, 3]. When the mathematical model of a plant that is to be controlled is completely known, then each of these approaches allows one to construct H_∞-controllers by using effective computational procedures that are available through MATLAB [4]. When the description of a plant involves uncertainties, the development of robust H_∞-controllers that provide a desired attenuation of disturbances, is reduced to the design of an H_∞-controller for some auxiliary system. The fully known system is derived from the initial uncertain system by a process of replacing the uncertainties by some additional artificial disturbances and some additional parameters. For systems with norm-bounded time-varying uncertainty parameters or for systems involving an unknown nonlinear static function (Lurie systems), these additional parameters act as Lagrange multipliers or, perhaps, as coefficients in an S-procedure [5, 6]. For systems described by transfer-function matrices with uncertain dynamic blocks, these parameters are the so-called scalings [7–10]. In either case, the construction of H_∞-controllers for the auxiliary problem is a very difficult task, mainly because the values of the additional parameters, for which appropriate Riccati equations or LMIs are feasible, are unknown *a priori*.

It was shown in [11–13] that the design of a robust H_∞- controller can be reduced to an optimization problem for some function that is subjected to LMI constraints. This function contains the additional parameters as variables. This optimization problem turns out to be convex in the case when the system state is available for measurement (state feedback case) [14, 15], but in the case when only an output of the system is available for measurement (output feedback case) this problem is non-convex in principle. Effective optimization algorithms would be helpful for tackling this problem. A few such algorithms were suggested in [12, 13]. The algorithm discussed in [16] uses the MATLAB command `mincx` to determine iteratively the minimum of a linear function that is subjected to LMI constraints. Almost all these algorithms involve essentially a search for two reciprocal matrices X and Y $(XY = I)$ that satisfy LMI constraints. The search for these matrices is based on minimizing some discrepancy of the kind $\Psi = \mathrm{trace}\,(X - Y^{-1})$. Formally, the conditions $XY = I$ and $\Psi = 0$ are equivalent. However, during the numerical process, Ψ can also vanish when $XY \neq I$ provided that X and Y^{-1} tend to zero. This is why it is very important to restrict the domain of admissible values of the variables.

Recently [17–23], it has been proved that the domain of admissible values of Lagrange multipliers, for which the H_∞-control problem for the auxiliary system is feasible, is bounded by a parallepiped whose boundaries have to be determined through a calculation process. This result is based on an analysis of Riccati equations. The limiting control possibilities allow one to derive an estimate for the robust performance of the original uncertain system [24]. Application of these theoretical results to the control of a parametrically disturbed pendulum shows that the corresponding parallepiped is fairly small. Hence, it is extremely difficult to determine admissible Lagrange parameters without having any knowledge of this parallelepiped.

In this paper, we combine the approaches described above concerning the design of a robust H_∞-controller and use an LMI-based optimization algorithm taking into account that scalings belong to the identified parallelepiped. The effectiveness of our algorithm is demonstrated by applying it to the problem of a parametrically disturbed pendulum.

2. Preliminary results

Consider a controlled plant

$$\dot{x} = Ax + B_1 v + B_2 u, \quad z = C_1 x + D_{12} u, \quad y = C_2 x + D_{21} v, \tag{1}$$

where x is a state, v is a disturbance input, u is a control input, z is a controlled output, y is a measured output and $A, B_1, B_2, C_1, C_2, D_{12}, D_{21}$ are given matrices of appropriate orders. The synthesis of an H_∞-controller for this plant comes down to deriving a dynamic output controller that provides attenuation of the disturbance in the given ratio γ, *i.e.*, providing fulfillment of the inequality

$$\sup_{v \neq 0} \frac{\|z\|}{\|v\|} < \gamma,$$

and internal stability of the closed-loop system, where

$$\|\eta\|^2 = \int_0^\infty |\eta(t)|^2 \, \mathrm{d}t,$$

for any vector function $\eta(t) \in L_2$.

It is well known [1] that under the following assumptions:

(i) (A, B_1) is stabilizable, (C_1, A) is detectable;
(ii) (A, B_2) is stabilizable, (C_2, A) is detectable;
(iii) $D_{12}^T (C_1 \, D_{12}) = (0 \, I)$;
(iv) $\begin{pmatrix} B_1 \\ D_{21} \end{pmatrix} D_{21}^T = \begin{pmatrix} 0 \\ I \end{pmatrix}$,

one of the possibilities, namely the so-called central H_∞-controller, is described by the equations

$$\dot{\hat{x}} = A_r \hat{x} + L_\infty (y - C_2 \hat{x}), \quad \hat{x}(0) = 0, \quad u = -\Theta \hat{x}, \quad \Theta = B_2^T X_\infty, \tag{2}$$

where $A_r = A + \gamma^{-2} B_1 B_1^T X_\infty - B_2 B_2^T X_\infty$, $L_\infty = Y_\infty (I - \gamma^{-2} X_\infty Y_\infty)^{-1} C_2^T$, and X_∞ and Y_∞ are semi-positive stabilizing solutions of the Riccati equations

$$A^T X + XA + X(\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X + C_1^T C_1 = 0 \tag{3}$$

and

$$AY + YA^T + Y(\gamma^{-2} C_1^T C_1 - C_2^T C_2) Y + B_1 B_1^T = 0, \tag{4}$$

provided that

$$\rho(X_\infty Y_\infty) < \gamma^2, \tag{5}$$

where $\rho(\cdot)$ denotes the spectral radius. The references [2,3] deal with an alternative characterization for H_∞-controllers in terms of LMIs. In particular, these controllers are defined by the equations (2), where $X_\infty = \gamma R^{-1}$ and $Y_\infty = \gamma S^{-1}$, and $R = R^T > 0, S = S^T > 0$ satisfy the following LMIs

$$\begin{pmatrix} RA^T + AR - \gamma B_2 B_2^T & B_1 & RC_1^T \\ B_1^T & -\gamma I & 0 \\ C_1 R & 0 & -\gamma I \end{pmatrix} < 0, \quad \begin{pmatrix} SA + A^T S - \gamma C_2^T C_2 & C_1^T & SB_1 \\ C_1 & -\gamma I & 0 \\ B_1^T S & 0 & -\gamma I \end{pmatrix} < 0, \tag{6,7}$$

$$\begin{pmatrix} R & I \\ I & S \end{pmatrix} \geq 0. \tag{8}$$

A necessary and sufficient condition for the existence of H_∞-controllers for given γ is the feasibility of these LMIs with respect to the variables $R > 0, S > 0$. It is easy to check this feasibility by means of the command `feasp` in MATLAB (toolbox LMI).

The synthesis of the optimal H_∞-controller, providing the minimal level of disturbance attenuation in the class Ξ of linear dynamic output controllers, requires finding the minimal H_∞-norm of the closed-loop system (1), *i.e.*,

$$\gamma_0 = \inf_{u \in \Xi} \sup_{v \not\equiv 0} \frac{\|z\|}{\|v\|}. \tag{9}$$

The search for this value can be reduced to a convex optimization problem, namely, to minimize γ under the convex constraints (6,7)–(8). This problem can be effectively tackled by using the command `mincx` in MATLAB.

3. Statement of the problem

We will consider uncertain systems described by Equations (1) with the matrix A of the form

$$A = A_0 + \sum_{k=1}^{n} F_k \Omega_k(t, x) E_k, \tag{10}$$

where A_0 is a given matrix, F_k are given matrices (the number of rows of each such matrix coincides with the dimension of the state vector, the number of columns being equal to $i_1, i_2, \ldots, i_n$, respectively), E_k are given matrices (the number of columns of each such matrix coincides with the dimension of the state vector and the number of rows is equal to $j_1, j_2, \dots, j_n$, respectively), and $\Omega_k(t,x)$ are unknown matrix functions satisfying the inequalities

$$\Omega_k(t,x)^T\Omega_k(t,x) \le I \;, \quad \forall t \ge 0, \quad \forall x, \quad k=1,\dots,n. \tag{11}$$

Such a structure of the matrix A allows us to describe uncertainty in different matrix entries and blocks. Denote a class of such uncertainties through Σ. The problem is to synthesize a linear dynamic output controller providing fulfillment of the inequality

$$\|z\| < \gamma\|v\|, \quad \forall v \in L_2, \quad v \not\equiv 0, \quad \forall \Omega_k(t,x) \in \Sigma, \tag{12}$$

with a minimally possible value of γ.

4. Auxiliary problem

The main idea of synthesizing a robust controller for the uncertain system under consideration is to replace the initial uncertain system by some auxiliary system without uncertainty, including additional disturbance inputs and additional parameters. This auxiliary system will be of the form

$$\dot{x} = A_0x + (B_1 \; \gamma\mu_1^{-1}F_1 \dots \gamma\mu_n^{-1}F_n)\xi + B_2u,$$
$$\hat{z} = \begin{pmatrix} C_1 \\ \mu_1E_1 \\ \vdots \\ \mu_nE_n \end{pmatrix} x + \begin{pmatrix} D_{12} \\ 0 \\ \vdots \\ 0 \end{pmatrix} u, \quad y = C_2x + (D_{21}\,0\dots0)\xi \;, \tag{13}$$

where $\xi = \mathrm{col}\,(\xi_0, \xi_1, \dots, \xi_n)$ is a disturbance input, $\hat{z}$ is a controlled output; all matrices in these equations are the same as those in Equations (1), (10), $\gamma > 0$ and $\mu_k > 0, k=1,\dots,n$ are some numbers.

In the particular case when $\xi_0 = v$ and $\xi_k = \gamma^{-1}\mu_k\,\Omega_k(t,x)E_kx, k=1,\dots,n$, Equation (13) coincides with Equation (1) with matrix A given in (10).

The control law that provides for the system (13) with some values of $\mu_k > 0, k=1\dots,n$ fulfillment of the inequality

$$\|\hat{z}\| < \gamma\|\xi\|, \quad \forall \xi \in L_2, \quad \xi \not\equiv 0, \tag{14}$$

also provides fulfillment of inequality (12) for the initial uncertain system (1), (10) with the same value of γ. Indeed, from (14) we obtain

$$\|z\|^2 + \sum_{k=1}^{n}\mu_k^2\|E_kx\|^2 < \gamma^2\|v\|^2 + \sum_{k=1}^{n}\mu_k^2\|\Omega_k(t,x)E_kx\|^2 \;.$$

Hence, taking into account (11), we have the inequality (12) is satisfied. Therefore, in what follows we will consider the H_∞-control problem for system (13).

5. Synthesis of robust H_∞-controllers based on Riccati equations

We will assume that the H_∞-control output problem for the auxiliary system (13) is feasible for a given γ and some values of μ_k. Since the assumptions (i)–(iv) are satisfied for this system, one possible robust H_∞-controller (the so-called central robust H_∞-controller) is described by

Equations (2) in which the appropriate matrices of the auxiliary system (13) enter. In this case, the Riccati equations (3) and (4) will be of the form

$$A_0^T X + X A_0 + X(\gamma^{-2} B_1 B_1^T + \sum_{k=1}^{n} \mu_k^{-2} F_k F_k^T - B_2 B_2^T) X + C_1^T C_1 + \sum_{k=1}^{n} \mu_k^2 E_k^T E_k = 0 \tag{15}$$

and

$$A_0 Y + Y A_0^T + Y(\gamma^{-2} C_1^T C_1 + \gamma^{-2} \sum_{k=1}^{n} \mu_k^2 E_k^T E_k - C_2^T C_2) Y + B_1 B_1^T + \gamma^2 \sum_{k=1}^{n} \mu_k^{-2} F_k F_k^T = 0. \tag{16}$$

As an upper estimate of the minimal possible value of γ in inequality (12) one may then use the minimal attenuation level in the auxiliary system. However, calculation of this value by means of the standard procedures `hinfric` or `hinflmi` of MATLAB, which allow to automatically find the minimum level γ_0 for a system without uncertainty, is impossible here because the matrices of the auxiliary system involve γ and the additional parameters μ_k.

In this situation, a search for such parameters μ_k that correspond to a minimal value of γ should be carried out. This search should be done in the domain where the H_∞-control problem for the auxiliary system is feasible. For given γ, the values of the parameters $\mu = (\mu_1, \ldots, \mu_n)$ for which the Riccati equations (15) and (16) for the auxiliary system (13) have stabilizing solutions satisfying (5) are referred to as admissible ones. Denote the domain of all admissible values of μ by $\Lambda_0(\gamma)$. The main difficulty of the above search is the *a priori* unboundedness of this domain. We will establish that, for a given γ, the domain $\Lambda_0(\gamma)$ is bounded and included within the boundaries of a parallelepiped which will be identified.

To this end, we write the Riccati equations (15) and (16) in standard form, using the additional parameters $\sigma_k, k = 1, \ldots, n$, defined by

$$\sigma_k = \gamma \mu_k^{-1}\ , \quad k = 1, \ldots, n\ . \tag{17}$$

Then the Equations (15) and (16) will be of the form

$$A_0^T X + X A_0 + X[\gamma^{-2} \hat{B}_1(\sigma) \hat{B}_1^T(\sigma) - B_2 B_2^T] X + \hat{C}_1^T(\mu) \hat{C}_1(\mu) = 0 \tag{18}$$

and

$$A_0 Y + Y A_0^T + Y[\gamma^{-2} \hat{C}_1^T(\mu) \hat{C}_1(\mu) - C_2^T C_2] Y + \hat{B}_1(\sigma) \hat{B}_1^T(\sigma) = 0\ , \tag{19}$$

where $\hat{B}_1(\sigma) = (B_1 \quad \sigma_1 F_1, \cdots, \sigma_n F_n)$ and $\hat{C}_1(\mu) = \mathrm{col}\,(C_1,\ \mu_1 E_1,\ \cdots,\ \mu_n E_n)$. Note that the above procedures of automatic search for the minimal γ can not also be applied to this problem due to the Equations (17).

The following statement based on results given in [18] is a necessary condition for the existence of a solution to the Riccati equation.

Lemma 1. *Let the Riccati equation*

$$\Psi^T P + P \Psi + P(\gamma^{-2} N_1 N_1^T - N_2 N_2^T) P + M^T M = 0\ , \tag{20}$$

where Ψ is a Hurwitz matrix, have a semi-positive stabilizing solution $P \geq 0$, i.e., such that $\Psi + (\gamma^{-2} N_1 N_1^T - N_2 N_2^T) P$ is Hurwitz. Then the following inequality

$$\gamma > \sup_{\omega} \sqrt{\rho(G(\omega))}\ , \tag{21}$$

holds, where

$$G(\omega)=H_1^T(-\mathrm{i}\,\omega)[I+H_2(\mathrm{i}\,\omega)H_2^T(-\mathrm{i}\omega)]^{-1}H_1(\mathrm{i}\,\omega), \tag{22}$$

$$H_1(s)=M(sI-\Psi)^{-1}N_1\ , \quad H_2(s)=M(sI-\Psi)^{-1}N_2\ .$$

Remark 1. From Lemma 1 it follows, if a stabilizing solition to the Riccati equation (20) exists, then

$$\gamma>\frac{\|H_1\|_\infty}{\sqrt{1+\|H_2\|_\infty^2}}\ , \tag{23}$$

where

$$\|H\|_\infty=\sup_\omega\sqrt{\rho(H^T(-\mathrm{i}\,\omega)H(\mathrm{i}\,\omega))}\ .$$

Actually, since

$$I+H_2(\mathrm{i}\,\omega)H_2^T(-\mathrm{i}\,\omega)\le I[1+\sup_\omega\rho(H_2^T(-\mathrm{i}\,\omega)H_2(\mathrm{i}\,\omega))]=I(1+\|H_2\|_\infty^2)\ ,$$

we obtain from (22)

$$G(\omega)\ge(1+\|H_2\|_\infty^2)^{-1}H_1^T(-\mathrm{i}\,\omega)H_1(\mathrm{i}\,\omega)$$

and, hence,

$$\sup_\omega\sqrt{\rho(G(\omega))}\ge\frac{\|H_1\|_\infty}{\sqrt{1+\|H_2\|_\infty^2}}\ . \tag{24}$$

Thus, we have (23).

From Lemma 1 it follows that a necessary condition for the existence of the required solutions to (18) and (19) will be determined by means of the functions

$$\Phi_1(\mu,\sigma)=\sup_\omega\sqrt{\rho(G_1(\omega,\mu,\sigma))}\ , \quad \Phi_2(\mu,\sigma)=\sup_\omega\sqrt{\rho(G_2(\omega,\mu,\sigma))}\ , \tag{25}$$

where $\mu=(\mu_1,\dots,\mu_n), \sigma=(\sigma_1,\dots,\sigma_n)$,

$$G_1(\omega,\mu,\sigma))=H_1^T(-\mathrm{i}\,\omega,\mu,\sigma)[I+H_2(\mathrm{i}\,\omega,\mu)H_2^T(-\mathrm{i}\omega,\mu)]^{-1}H_1(\mathrm{i}\,\omega,\mu,\sigma)\ , \tag{26}$$

$$G_2(\omega,\mu,\sigma))=\hat{H}_1^T(-\mathrm{i}\,\omega,\mu,\sigma)[I+\hat{H}_2(\mathrm{i}\,\omega,\sigma)\hat{H}_2^T(-\mathrm{i}\omega,\sigma)]^{-1}\hat{H}_1(\mathrm{i}\,\omega,\mu,\sigma)\ ,$$

$$H_1(s,\mu,\sigma)=\hat{C}_1(\mu)(sI-A_0)^{-1}\hat{B}_1(\sigma), \quad H_2(s,\mu)=\hat{C}_1(\mu)(sI-A_0)^{-1}B_2\ , \tag{27}$$

$$\hat{H}_1(s,\mu,\sigma)=\hat{B}_1^T(\sigma)(sI-A_0^T)^{-1}\hat{C}_1^T(\mu)\ , \quad \hat{H}_2(s,\sigma)=\hat{B}_1^T(\sigma)(sI-A_0^T)^{-1}C_2^T \tag{28}$$

according to the next statement.

Theorem 1. *Let for the system* (13), *with a Hurwitz matrix A_0, the assumptions $D_{12}^T(C_1\,D_{12})=(0\,I), D_{21}(B_1^T\,D_{21}^T)=(0\,I)$ hold. Then the domain $\Lambda_0(\gamma)$ of admissible values of the parameters μ belongs to the domain $\Lambda_1(\gamma)$ defined by*

$$\Lambda_1(\gamma)=\{\mu:\max\{\Phi_1(\mu_1,\dots,\mu_n,\gamma\mu_1^{-1},\dots,\gamma\mu_n^{-1}),$$
$$\Phi_2(\mu_1,\dots,\mu_n,\gamma\mu_1^{-1},\dots,\gamma\mu_n^{-1})\}<\gamma\}. \tag{29}$$

In the general case, the domain $\Lambda_1(\gamma)$ does not coincide with the domain $\Lambda_0(\gamma)$, since Lemma 1 provides a necessary condition only and, moreover, the condition (5) is not taken into account. From a computational point of view, the description of the domain $\Lambda_1(\gamma)$ is rather complicated, since for finding the functions $\Phi_j(\mu,\sigma), j=1,2$, an optimization procedure of the spectral radius of matrix functions over ω is needed.

It turns out that the domain $\Lambda_1(\gamma)$ is included in an open parallelepiped the boundaries of which are easily calculated.

Theorem 2. *Let for the system* (13) *in which* A_0 *is Hurwitz and* $D_{12}^T(C_1\,D_{12}) = (0\,I)$, $D_{21}(B_1^T\,D_{21}^T) = (0\,I)$, *the problem of output* H_∞*-control for a given* γ *and some* $\mu = (\mu_1, \dots, \mu_n)$ *be feasible. Then the values of* μ *belong to the open parallelepiped*

$$\Pi(\gamma) = \{\mu_k \in (\mu_k^-,\ \mu_k^+\gamma),\ k = 1, \dots, n\}\ , \tag{30}$$

$$\mu_k^- = \frac{\|Q_k\|_\infty}{\sqrt{1 + \|H_{12}\|_\infty^2}}\ , \quad \mu_k^+ = \frac{\sqrt{1 + \|H_{21}\|_\infty^2}}{\|R_k\|_\infty}\ , \tag{31}$$

and

$$\gamma > \gamma_* = \max\{\gamma_1, \dots, \gamma_k, \dots, \gamma_n, \gamma_{12}, \gamma_{21}\}\ , \tag{32}$$

$$\gamma_k = \frac{\|Q_k\|_\infty \|R_k\|_\infty}{\sqrt{(1 + \|H_{12}\|_\infty^2)(1 + \|H_{21}\|_\infty^2)}}\ , \quad \gamma_{12} = \frac{\|H_{11}\|_\infty}{\sqrt{1 + \|H_{12}\|_\infty^2}}\ , \quad \gamma_{21} = \frac{\|H_{11}\|_\infty}{\sqrt{1 + \|H_{21}\|_\infty^2}}\ ,$$

$$H_{11}(s) = C_1(sI - A_0)^{-1}B_1\ , \quad H_{12}(s) = C_1(sI - A_0)^{-1}B_2\ , \quad H_{21}(s) = C_2(sI - A_0)^{-1}B_1\ ,$$

$$Q_k(s) = C_1(sI - A_0)^{-1}F_k\ , \quad R_k(s) = E_k(sI - A_0)^{-1}B_1\ .$$

Note that the lower bounds μ_k^- of the parallelepiped $\Pi(\gamma)$ do not depend on γ, while its upper bounds, which are equal to $\mu_k^+\gamma$, grow linearly with γ. Moreover, $\Pi(\gamma)$ will be bounded and, hence, the domain of admissible values of μ will be bounded as well, provided that $\|R_k\|_\infty \neq 0$ for all $k = 1, \dots, n$.

From Theorem 2 it follows that, for a given γ, the domain of admissible values of μ is included in the bounded parallelepiped $\Pi(\gamma)$. Outside of this parallelepiped the H_∞-control problem for the auxiliary system is not feasible. Thus, to estimate the minimally possible γ, one should create a finite grid in $\Pi(\gamma)$, check the feasibility of the H_∞-control problem at all of its points by means of a standard MATLAB command and find the minimum value of γ for which this problem is feasible.

Remark 2. In view of the estimate given in (24), the admissible values of μ satisfy the condition

$$\max\left\{\frac{\|H_1\|_\infty}{\sqrt{1 + \|H_2\|_\infty^2}}, \frac{\|\hat{H}_1\|_\infty}{\sqrt{1 + \|\hat{H}_2\|_\infty^2}}\right\} < \gamma\ , \tag{33}$$

where H_1 and H_2 are derived from $H_1(s, \mu, \alpha)$ and $H_2(s, \mu, \alpha)$ given in (27), and $\hat{H}_1$ and $\hat{H}_2$ are obtained from $\hat{H}_1(s, \mu, \alpha)$ and $\hat{H}_2(s, \mu, \alpha)$ given in (28) for $\alpha_k = \gamma\mu_k^{-1}, k = 1, \dots, n$. Thus, failure to execute the inequality (33) in some point of the grid means that this point does not belong to $\Lambda_0(\gamma)$ and, consequently, it is not required to check feasibility of the H_∞-control problem in this point.

The computational procedure for finding the upper estimate of the minimal possible value of γ involves the following steps:

1. using the MATLAB procedure normhinf, calculate the values of $\gamma_*, \mu_k^-, \mu_k^+, k=1,\ldots,n$;
2. for a given $\gamma > \gamma_*$, find a finite set of values of μ in the parallelepiped $\Pi(\gamma)$ for which the existence of stabilizing solutions (satisfying condition (5) of the Riccati equations (15) and (16) is numerically checked with the help of MATLAB;
3. if at least one such point exists, then the given γ is an upper estimate for the minimum possible value of γ; to adjust it, one must decrease the value of γ and go to step 2;
4. if there is no such a point, one must increase γ and go to step 2.

After a finite number of steps of this procedure, two values γ_l and γ_u will be found such that, for γ_l, the problem of output H_∞-control for the auxiliary system is infeasible for any positive values of μ_k, while, for γ_u, there exist values μ_k for which the problem is feasible. The existence of such a γ_u is ensured by the assumption that the problem of output H_∞-control for the auxiliary system is feasible. Using the bisection algorithm for the interval $[\gamma_l, \gamma_u]$, we obtain two sequences of the values $\gamma_l^{(i)}$ and $\gamma_u^{(i)}$ representing at the ith iteration the lower and upper bounds for the upper estimate of the minimal possible value of γ for the initial uncertain system (1), (10). We stop this process when the condition $\gamma_u^{(i)} - \gamma_l^{(i)} < \varepsilon$ holds for a given ε. At this iteration, let $\gamma_u = \bar{\gamma}$ and $\bar{\mu}_k$ be the corresponding values of the parameters μ_k for which the problem of output H_∞-control for the auxiliary system is feasible. Then, according to (2), the required robust controller, which attenuates the disturbance in the initial uncertain system in the ratio of no more than $\bar{\gamma}$, is described by the following equations

$$\begin{aligned} &\dot{\hat{x}} = A_r\hat{x} + \bar{L}(y - C_2\hat{x}) \ , \quad A_r = A_0 + \bar{\gamma}^{-2}\hat{B}_1(\bar{\alpha})\hat{B}_1^T(\bar{\alpha})\bar{X} - B_2B_2^T\bar{X} \ , \\ &u = -\Theta\hat{x} \ , \quad \Theta = B_2^T\bar{X} \ , \quad \hat{x}(0) = 0 \ , \end{aligned} \tag{34}$$

where $\bar{L} = \bar{Y}(I - \bar{\gamma}^{-2}\bar{X}\bar{Y})^{-1}C_2^T$, $\bar{X}$ and $\bar{Y}$ are semi-positive stabilizing solutions to the Riccati equations (18) and (19), with $\mu = (\bar{\mu}_1,\ldots,\bar{\mu}_n), \alpha = \bar{\alpha} = (\bar{\alpha}_1,\ldots,\bar{\alpha}_n), \bar{\alpha}_k = \bar{\gamma}\bar{\mu}_k^{-1}$, $k=1,\ldots,n$.

6. Synthesis of robust H_∞-controllers based on linear matrix inequalities

Consider an alternative approach to the synthesis of robust H_∞-controllers based on solving LMIs. Represent the Equations (13) of the auxiliary system in the form

$$\begin{aligned} &\dot{x} = A_0x + (B_1\ \gamma\alpha_1^{1/2}F_1 \ldots \gamma\alpha_n^{1/2}F_n)\xi + B_2u \\ &\hat{z} = \begin{pmatrix} C_1 \\ \alpha_1^{-1/2}E_1 \\ \vdots \\ \alpha_n^{-1/2}E_n \end{pmatrix} x + \begin{pmatrix} D_{12} \\ 0 \\ \vdots \\ 0 \end{pmatrix} u \\ &y = C_2x + (D_{21}\,0\ldots 0)\xi \ . \end{aligned} \tag{35}$$

Introduce

$$F = (F_1\,F_2\,\ldots F_n) \ , \quad E^T = (E_1^T\,E_2^T\,\ldots E_n^T)$$

and the following diagonal matrices

$$V_1 = \mathrm{diag}\,(\alpha_1 I_{i_1}, \alpha_2 I_{i_2}, \ldots, \alpha_n I_{i_n}) \ , \quad V_2 = \mathrm{diag}\,(\alpha_1 I_{j_1}, \alpha_2 I_{j_2}, \ldots, \alpha_n I_{j_n}) \ ,$$

where I_k is the unit $k\times k$-matrix. Then an H_∞-controller for the auxiliary system exists if the following LMIs corresponding to (6,7)–(8) are feasible

$$\Phi_1(R,\alpha_1,\dots,\alpha_n)=\begin{pmatrix} RA_0^T+A_0R-\gamma B_2B_2^T & B_1 & FV_1 & RC_1^T & RE^T \\ B_1^T & -\gamma I & 0 & 0 & 0 \\ V_1F^T & 0 & -V_1 & 0 & 0 \\ C_1R & 0 & 0 & -\gamma I & 0 \\ ER & 0 & 0 & 0 & -V_2 \end{pmatrix}<0\ , \tag{36}$$

$$\Phi_2(S,\alpha_1^{-1},\dots,\alpha_n^{-1})=\begin{pmatrix} SA_0+A_0^TS-\gamma C_2^TC_2 & C_1^T & E^TV_2^{-1} & SB_1 & SF \\ C_1 & -\gamma I & 0 & 0 & 0 \\ V_2^{-1}E & 0 & -V_2^{-1} & 0 & 0 \\ B_1^TS & 0 & 0 & -\gamma I & 0 \\ F^TS & 0 & 0 & 0 & -V_1^{-1} \end{pmatrix}<0\ , \tag{37}$$

$$\Phi_3(R,S)=\begin{pmatrix} R & I \\ I & S \end{pmatrix}\ge 0\ . \tag{38}$$

Now these are LMIs with respect to the variables R and S for fixed values of $\gamma,\alpha_1,\alpha_2,\dots,\alpha_n$. To check the feasibility of these LMIs, one should carry out an enumeration of the possibilities in some domain of the space of the variables $\gamma,\alpha_1,\alpha_2,\dots,\alpha_n$. However, this is a rather complicated procedure.

Note here that the synthesis of the robust state feedback H_∞-controller is reduced to solving the following LMIs

$$R>0\ ,\quad \alpha_1>0\ ,\dots\ ,\alpha_n>0\ ,\quad \Phi_1(R,\alpha_1,\dots,\alpha_n)<0\ .$$

Moreover, since γ enters linearly in (36), the problem of optimal (with minimal possible value of γ) robust state feedback H_∞-control is reduced to the following convex optimization problem

$$\min\{\gamma:\gamma>0,\quad R>0,\quad \alpha_1>0\ ,\dots\ ,\alpha_n>0\ ,\quad \Phi_1(R,\alpha_1,\dots,\alpha_n)<0\}\ ,$$

which can be effectively tackled via the command `mincx` in MATLAB.

In the general case of the output feedback problem, robust H_∞-control synthesis is not reduced to a convex optimization problem. Nevertheless, we will try to solve this problem by reducing the non-convex terms and then using a technique of convex optimization. To this end, note that the variables $\alpha_1,\dots,\alpha_n$ enter into the first inequality linearly, while the variables $\alpha_1^{-1},\dots,\alpha_n^{-1}$ enter into the second inequality linearly. Introduce additional variables

$$W_1=\mathrm{diag}\,(\beta_1I_{i_1},\beta_2I_{i_2},\dots,\beta_nI_{i_n})\ ,\quad W_2=\mathrm{diag}\,(\beta_1I_{j_1},\beta_2I_{j_2},\dots,\beta_nI_{j_n})$$

and replace V_1^{-1} and V_2^{-1} by W_1 and W_2, respectively, in the inequality (37). Then we write a new LMI

$$\Phi_2(S,\beta_1,\dots,\beta_n)=\begin{pmatrix} SA_0+A_0^TS-\gamma C_2^TC_2 & C_1^T & E^TW_2 & SB_1 & SF \\ C_1 & -\gamma I & 0 & 0 & 0 \\ W_2E & 0 & -W_2 & 0 & 0 \\ B_1^TS & 0 & 0 & -\gamma I & 0 \\ F^TS & 0 & 0 & 0 & -W_1 \end{pmatrix}<0\ . \tag{39}$$

Now, we have to provide the equalities

$$\alpha_k \beta_k = 1 \ , \quad k = 1, 2, \dots, n \ , \tag{40}$$

which are equivalent to two groups of inequalities

$$\alpha_k \beta_k \geq 1 \ , \quad \alpha_k \beta_k \leq 1 \ , \quad k = 1, 2, \dots, n \ .$$

The first of these can be rewritten as LMIs

$$\Psi_k(\alpha_k, \beta_k) = \begin{pmatrix} \alpha_k & 1 \\ 1 & \beta_k \end{pmatrix} \geq 0 \ , \quad k = 1, 2, \dots, n \ , \tag{41}$$

while the second can not be presented as LMIs in principle, since these inequalities define a non-convex domain in the space of α_k, β_k.

Taking this fact into account, we introduce the function

$$\Psi(\alpha, \beta) = \sum_{k=1}^{n} (\alpha_k - \beta_k^{-1}) \ ,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = \beta_1, \dots, \beta_n$.

According to Theorem 2 we have

$$(\mu_k^-)^2 \leq \beta_k \leq (\mu_k^+)^2 \gamma^2 \ , \quad k = 1, 2, \dots, n \ . \tag{42}$$

Denote by Λ the set of variables $R, S, \alpha_k, \beta_k, k = 1, \dots, n$, satisfying (42) and the following LMIs

$$\begin{aligned} & R > 0, \quad S > 0, \quad \alpha_k > 0 \ , \\ & \Phi_1(R, \alpha) < 0, \quad \Phi_2(S, \beta) < 0, \\ & \Phi_3(R, S) \geq 0, \quad \Psi_k(\alpha_k, \beta_k) \geq 0 \ , \quad k = 1, \dots, n. \end{aligned} \tag{43}$$

Now, formulate the following optimization problem: find a minimum of the function $\Psi(\alpha, \beta)$ on the set Λ. If this minimum is equal to zero then, taking into account the boundedness of the variables β_k, we obtain the desired equalities (40). Thus, we arrive at the following statement.

Theorem 3. *Suppose that for a given γ the equality*

$$\inf\{\Psi(\alpha, \beta) : R, S, \alpha_k, \beta_k \ (k = 1, \dots, n) \in \Lambda\} = 0 \tag{44}$$

holds. Then the robust H_∞-control problem for the uncertain system (1), (10) *is feasible for this γ.*

The main difficulty when solving this problem is that the function to be minimized is not convex and, consequently, a search for its global minimum can not be conducted by convex optimization procedures. For this problem we suggest an optimization algorithm that involves an iterative procedure at each step of which a minimum of a linear function subject to LMI constraints is found.

Introduce the following auxiliary function

$$\hat{\Psi}(\alpha, \beta, q) = \sum_{k=1}^{n} (\alpha_k + q_k^2 \beta_k + 2q_k) \ ,$$

where $q = (q_1, \dots, q_n)$. This is a linear function in the variables $\alpha_k, \beta_k, k = 1, \dots, n$, and for $q_k = -\beta_k^{-1}, k = 1, \dots, n$, we have $\hat{\Psi}(\alpha, \beta, q) = \Psi(\alpha, \beta)$. Fix values of the variables $q = q^{(0)}$, *i.e.*, $q_k = q_k^{(0)}, k = 1, \dots, n$, and, instead of the optimization problem (44), consider the following optimization problem for $q = q^{(0)}$:

$$\min\{\hat{\Psi}(\alpha,\beta,q): R, S, \alpha_k, \beta_k\ (k=1,\ldots,n)\in\Lambda\}\ . \tag{45}$$

The minimum of this function is calculated numerically via the procedure `mincx`. Note that the LMIs $\Phi_3(R,S)\geq 0, \Psi_k(\alpha_k,\beta_k)\geq 0\ ,\quad k=1,\ldots,n$ in (43) are presented as strict ones due to certain features of the toolbox LMI. Let it be reached at the following values $\alpha^{(0)}=(\alpha_1^{(0)},\ldots,\alpha_n^{(0)})$ and $\beta^{(0)}=(\beta_1^{(0)},\ldots,\beta_n^{(0)})$. Replace the values of the variables q : $q_k^{(1)}=-1/\beta_k^{(0)}, k=1,\ldots,n$ and solve the optimization problem (45) when $q=q^{(1)}$. Let its minimum be reached at $\alpha^{(1)}=(\alpha_1^{(1)},\ldots,\alpha_n^{(1)}), \beta^{(1)}=(\beta_1^{(1)},\ldots,\beta_n^{(1)})$. Then choose $q_k^{(2)}=-1/\beta_k^{(1)}, k=1,\ldots,n$ and continue the above process. The process is assumed to be terminated when at least one of the following two inequalities

$$\eta(\alpha^{(j)},\beta^{(j)})<\varepsilon$$

or

$$|\Delta\eta^{(j)}|=|\eta(\alpha^{(j+1)},\beta^{(j+1)})-\eta(\alpha^{(j)},\beta^{(j)})|<\varepsilon$$

holds, where

$$\eta(\alpha,\beta)=\sum_{k=1}^{n}(\alpha_k\beta_k-1)$$

and $\varepsilon>0$ is a given accuracy.

The above algorithm possesses the following important property.

Theorem 4. *For any initial values* $q^{(0)}$,

$$\lim_{j\to\infty}\Psi(\alpha^{(j)},\beta^{(j)})=\Psi_\infty\geq 0\ ,$$

and the algorithm comes to stop in a finite number of steps.

From Theorem 4 it follows that the algorithm may result in one of two possible situations. If $\eta_\infty=0$, the auxiliary H_∞-control problem and, consequently, the robust H_∞-control problem for uncertain system (1), (10) due to Theorem 3 is feasible for the given γ. If $\eta_\infty>0$, one cannot make a decision regarding the feasibility of the auxiliary problem defined above since the convergence of this algorithm to the global minimum of the function $\Psi(\alpha,\beta)$ is not guaranteed. In the latter situation, it is recommended, for example, to repeat the above process for other initial values $q^{(0)}$, as is usually done in global optimization.

The search for the minimum value of γ is carried out, as usual, by means of the bisection algorithm. The efficiency of the proposed algorithm will be demonstrated in the next section.

7. Optimal attenuation of oscillations for a parametrically disturbed pendulum

Consider a controlled pendulum under parametric and external disturbances described by the equation

$$\begin{aligned}&\ddot{\varphi}+[1+f_1\Omega_1(t,\varphi,\dot{\varphi})]\dot{\varphi}+\omega_0^2[1+f_2\Omega_2(t,\varphi,\dot{\varphi})]\varphi=u+v_1,\\ &\psi=\varphi+v_2\ ,\quad \varphi(0)=\dot{\varphi}(0)=0,\end{aligned} \tag{46}$$

where $\omega_0,\ f_i, i=1,2(\omega_0\neq 0, 0\leq f_i<1)$ are given parameters, u is the control input, v_1 is the external disturbance, v_2 is the measurement disturbance. The functions $\Omega_i(t,\varphi,\dot{\varphi})$ correspond to parametric disturbances and satisfy the inequalities

$$|\Omega_i(t,\varphi,\dot{\varphi})|\leq 1\ ,\quad \forall t,\ \varphi,\ \dot{\varphi}\ ,\quad i=1,2. \tag{47}$$

Denote a class of such functions by Σ. The disturbances $v^T = (v_1, v_2)$ are assumed to be satisfied the inequality

$$J_1(v) = \int_0^\infty [v_1^2(t) + v_2^2(t)]\,\mathrm{d}t < \infty \ .$$

The performance index is chosen as

$$J_2(u, v) = \int_0^\infty (\omega_0^2\varphi^2 + \dot{\varphi}^2 + u^2)\,\mathrm{d}t \ ,$$

where the integrand is related to the mechanical energy of the undisturbed pendulum plus control costs. It is required to synthesize a controller using the measured output ψ and providing the optimal attenuation of the external and measurement disturbances, *i.e.*, the inequality

$$\frac{J_2(u, v)}{J_1(v)} < \gamma^2 \quad \forall v \in L_2, \quad v \not\equiv 0 \ , \quad \forall \Omega_i(t, \varphi, \dot{\varphi}) \in \Sigma, \ i = 1, 2, \tag{48}$$

is to be fulfilled with the minimum value of γ under any admissible parametric disturbances.

Let us rewrite the pendulum equation in the form (1), (10), in which $x = \mathrm{col}\,(\varphi, \dot{\varphi})$, $v = \mathrm{col}\,(v_1, v_2)$, $y = \psi$,

$$A_0 = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} \omega_0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$D_{12} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 \\ f_1 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 \\ f_2 \end{pmatrix},$$

$$C_2 = (1 \ 0), \quad D_{21} = (0 \ 1), \quad E_1 = (0 \ -1), \quad E_2 = (-\omega_0^2 \ 0).$$

The pendulum parameters are chosen as follows: $\omega_0 = 10$ and $f_1 = f_2 = 0{\cdot}1$.

By using the synthesis based on Riccati equations, we obtain

$$\gamma_1 = 0{\cdot}0812, \quad \gamma_2 = 0{\cdot}8126, \quad \gamma_{12} = 0{\cdot}8165, \quad \gamma_{21} = 1{\cdot}407.$$

According to (32) this means that $\gamma_* = 1{\cdot}407$ and the parallelepiped (30) is given by

$$0{\cdot}0817 < \mu_1 < 1{\cdot}0055\gamma, \quad 0{\cdot}0817 < \mu_2 < 0{\cdot}1005\gamma.$$

The algorithm results in $\bar{\mu}_1 = 0{\cdot}3856, \bar{\mu}_2 = 0{\cdot}1395, \gamma_l = 15{\cdot}3$ and $\gamma_u = \bar{\gamma} = 15{\cdot}5$. Thus, the controller (34), where

$$A_r = \begin{pmatrix} 0 & 1 \\ -100{\cdot}61 & -1{\cdot}6419 \end{pmatrix}, \quad \bar{L} = \begin{pmatrix} 9{\cdot}316 \\ 34{\cdot}117 \end{pmatrix}, \quad \Theta^T = \begin{pmatrix} 1{\cdot}469 \\ 1{\cdot}547 \end{pmatrix},$$

provides attenuating oscillations for the parametrically disturbed pendulum in a ratio of no more than 15·5.

By using the synthesis based on LMIs, we obtained the following results. The initial parameters of the algorithm were $q_1^{(0)} = q_2^{(0)} = 1$. Calculations in MATLAB were carried out with accuracy 10^{-6} and $\varepsilon = 10^{-5}$. For $\gamma < 15{\cdot}303$, the corresponding auxiliary problem was not feasible. For $\gamma = 15{\cdot}303$, there were only two iterations: on the first iteration

$$\alpha_1^{(0)} = 6{\cdot}0318, \quad \alpha_2^{(0)} = 60{\cdot}5862, \quad \beta_1^{(0)} = 0{\cdot}1658, \quad \beta_2^{(0)} = 0{\cdot}0166$$

and $\eta = 0{\cdot}0043$; while on the second iteration

$$\alpha_1^{(1)} = 6{\cdot}0627, \quad \alpha_2^{(1)} = 60{\cdot}6532, \quad \beta_1^{(1)} = 0{\cdot}1649, \quad \beta_2^{(1)} = 0{\cdot}0165$$

and $\eta = 1{\cdot}07 \times 10^{-6}$, which, in view of the given accuracy, means that the control problem is feasible. The parameters of the robust H_∞-controller (2) with performance 15·303 are as follows

$$A_r = \begin{pmatrix} 0 & 1 \\ -100{\cdot}4450 & -1{\cdot}5128 \end{pmatrix}, \quad L_\infty = \begin{pmatrix} 0{\cdot}1686 \\ 1{\cdot}5480 \end{pmatrix} 10^7, \quad \Theta^T = \begin{pmatrix} 1{\cdot}3215 \\ 1{\cdot}5229 \end{pmatrix}.$$

Note that the different approaches presented above result in different robust H_∞-controllers providing attenuating levels close each other. For comparison, note that the minimal H_∞-performance for a pendulum without parametric disturbances is equal to 1·41.

8. Conclusion

Two main approaches to robust H_∞-controller design for systems with time-varying norm-bounded uncertainties based on solving algebraic Riccati equations and on solving linear matrix inequalities have been considered.

The problem of synthesizing an optimal robust controller for uncertain systems is solved through embedding into an auxiliary completely known system with scalings. To solve the latter problem, it is required to find admissible values of the scalings such that the corresponding Riccati equations and LMIs have appropriate solutions and to search for a minimum attenuating level of disturbances in the whole domain of admissible values.

It has been shown that the domain of admissible values of the scalings is bounded and inscribed in a parallelepiped whose boundaries can easily be determined using MATLAB procedures. The presented example of a controlled parametrically disturbed pendulum demonstrates that the domain of admissible values can be very small and, if there is no data on it, the search for even a single admissible point in the space of scalings becomes very complicated.

An optimization algorithm for checking the feasibility of the robust H_∞-control problem using an LMI technique and the boundaries of the admissible values for the scalings has been suggested. This algorithm is implemented by means of the standard minimization procedure for a linear function under LMI constraints (command `mincx` in the LMI toolbox). It has been shown that the sequence of the function values generated by the algorithm is always converging. If its limiting value is equal to zero, the original robust H_∞-control problem is feasible. The main difficulty in the implementation of this approach is to make a decision about the problem feasibility when the limiting value turns out to be nonzero. In latter case, one can, for example, repeat the above process for other initial values of the algorithm, as is usually done in global optimization. A numerical example of optimal oscillation attenuation for a parametrically disturbed pendulum illustrates the effectiveness of the proposed algorithm.

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